Integer Linear Programming
CSCI 532

Announcements

- No Class on April 2nd.

- HW 3 grading.

- Final Presentation planning.

x_i **is 0 or 1, not 0.62**

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What is this for?

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What have we accomplished?

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What have we accomplished?

Solving ILPs is NP-Hard.

Questions?

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What have we accomplished?

Solving ILPs is NP-Hard.

In general

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

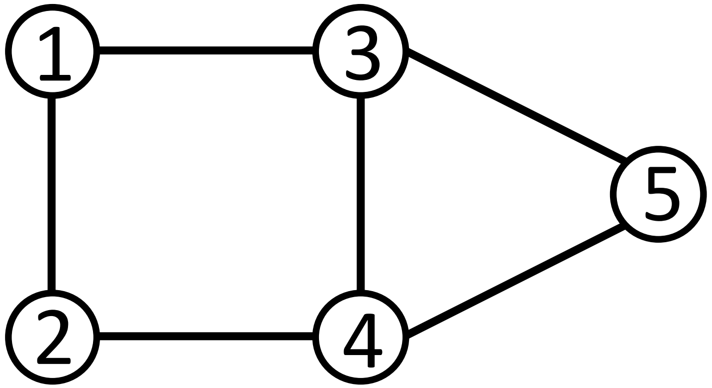
Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:

$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$



Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

<u>Example:</u>

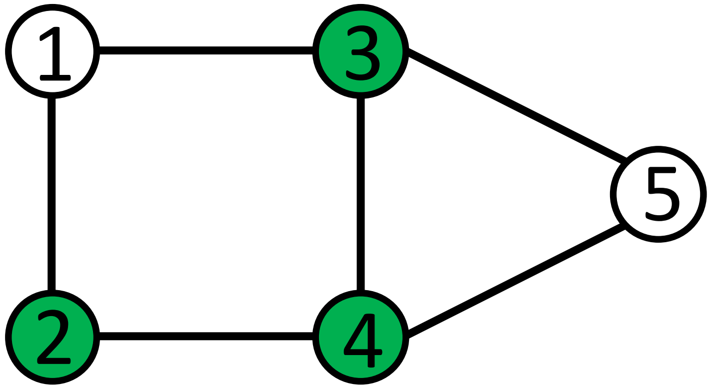
Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

Optimal Solution:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$

Objective = 3



Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

What happens?

$x_i \in [0,1]$ = Indicates if vertex i is selected.

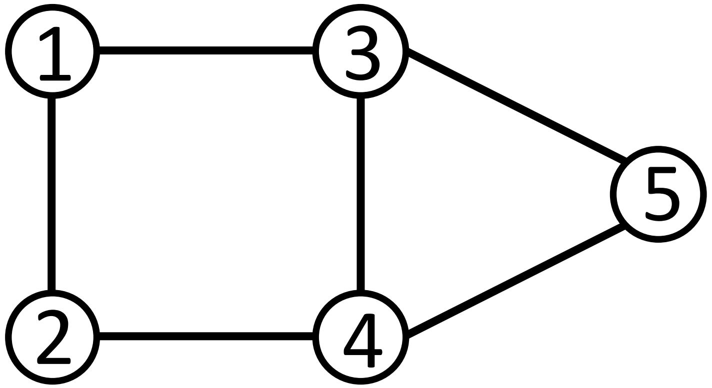
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$



Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$, find the smallest $V' \subseteq V$ such that each edge in E contains an end point in V'?

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

What happens?

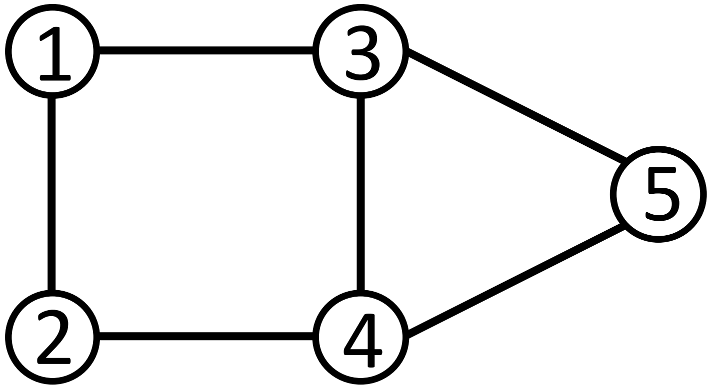
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

Objective = 2.5

<u>Example:</u>

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$



ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Optimal:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$

Objective = 3

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Optimal:

$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

Objective = 2.5

ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Optimal:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$

Objective = 3

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Optimal:

$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

Objective = 2.5

Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem).

ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.	
Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$
Objective = 3

$x_i \in [0,1]$ = Indicates if vertex i is selected.	
Objective:	$\min \sum_i x_i$
Subject to:	$x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal:

$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$
Objective = 2.5

Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem). If the minimum objective value comes from an integer solution, a plain LP solver (e.g., Simplex) will find it.

$x_1, x_2 \in \mathbb{R}$

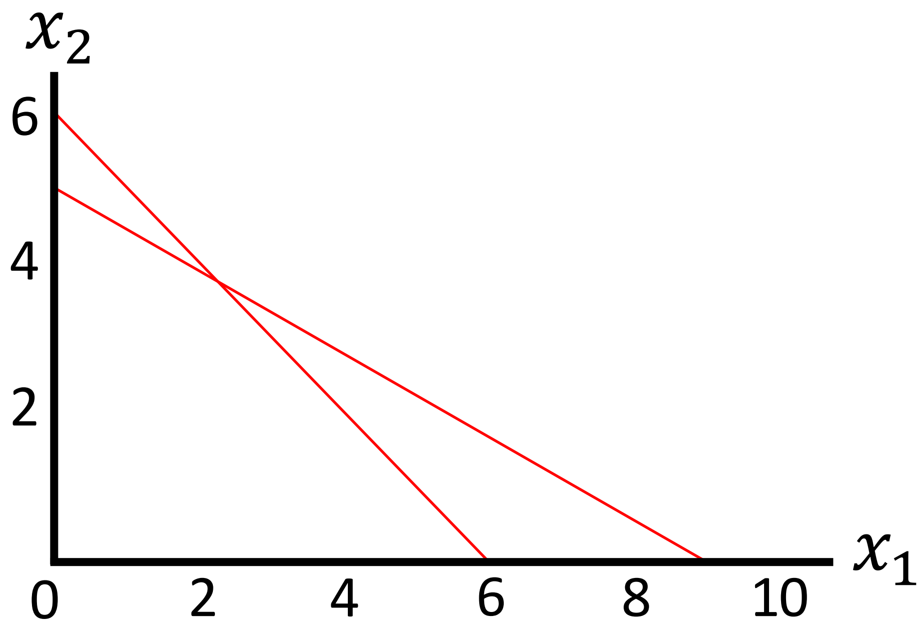
Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Why are ILPs hard to solve?

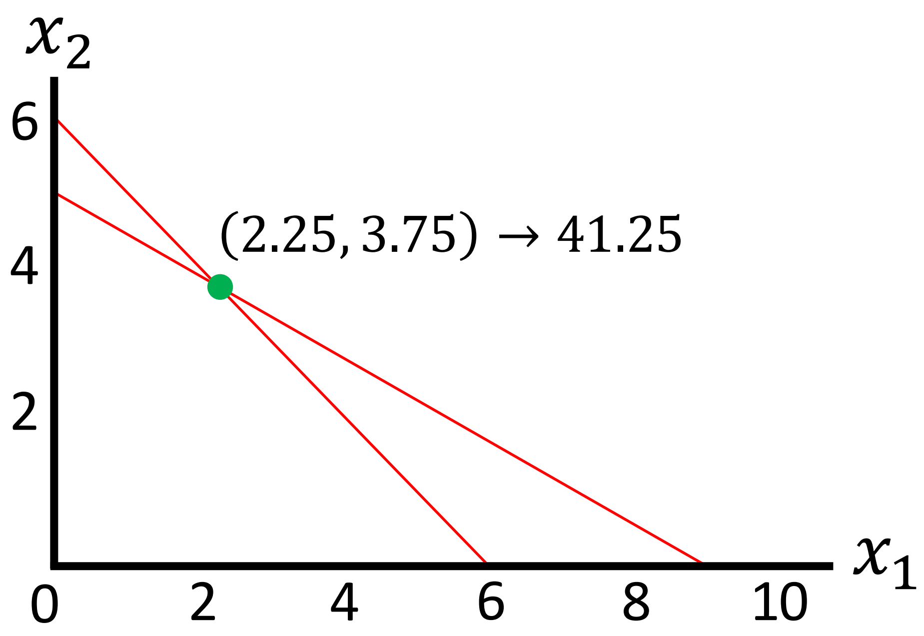


$x_1, x_2 \in \mathbb{R}$

Objective: $\max 5x_1 + 8x_2$

Subject to:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_1, x_2 \geq 0$

Why are ILPs hard to solve?



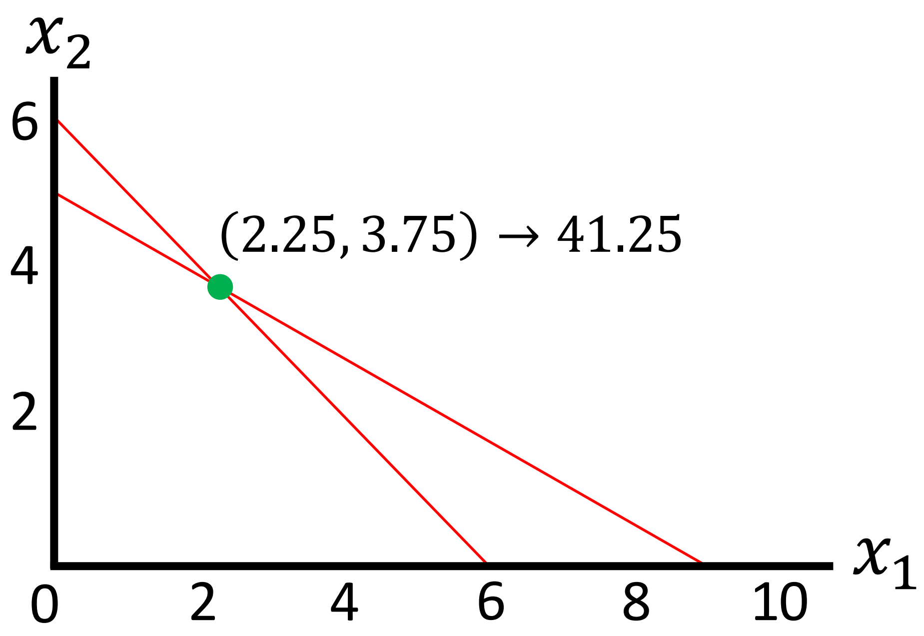
$x_1, x_2 \in \mathbb{R}$

Objective: $\max 5x_1 + 8x_2$

Subject to:

$$x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, x_2 \geq 0$$

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

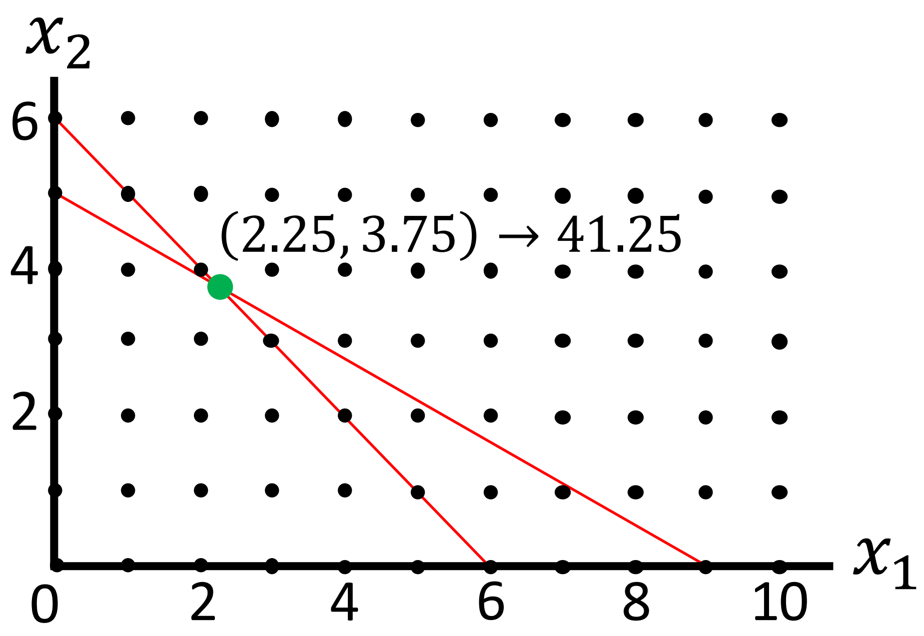
Subject to:

$$x_1 + x_2 \leq 6$$

$$5x_1 + 9x_2 \leq 45$$

$$x_1, x_2 \geq 0$$

Why are ILPs hard to solve?

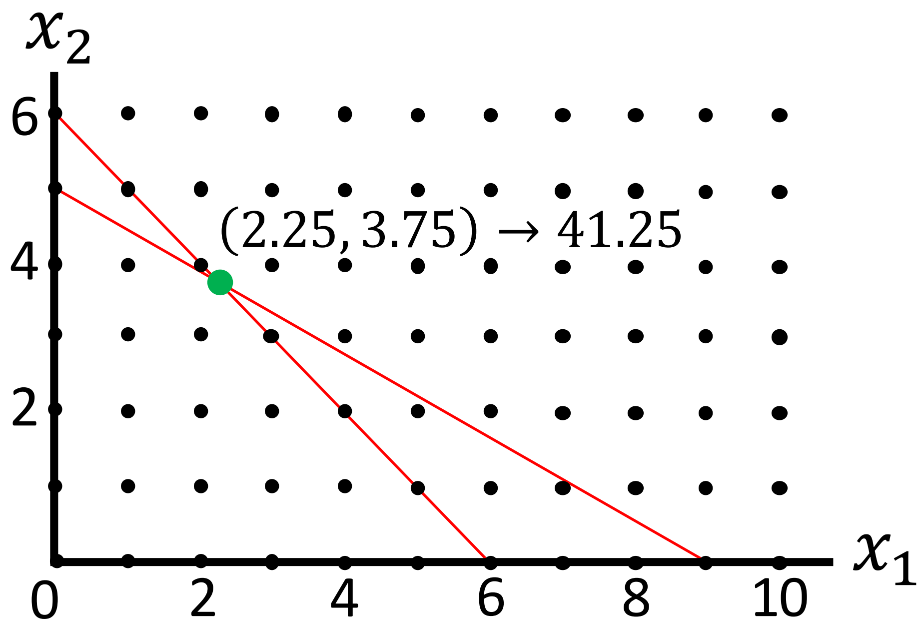


$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_1, x_2 \geq 0$

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

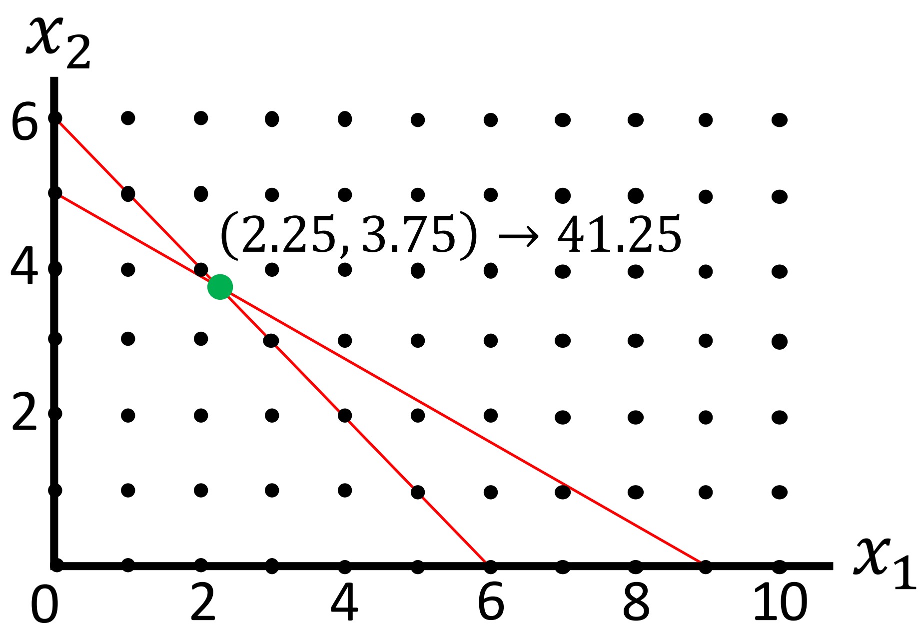
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

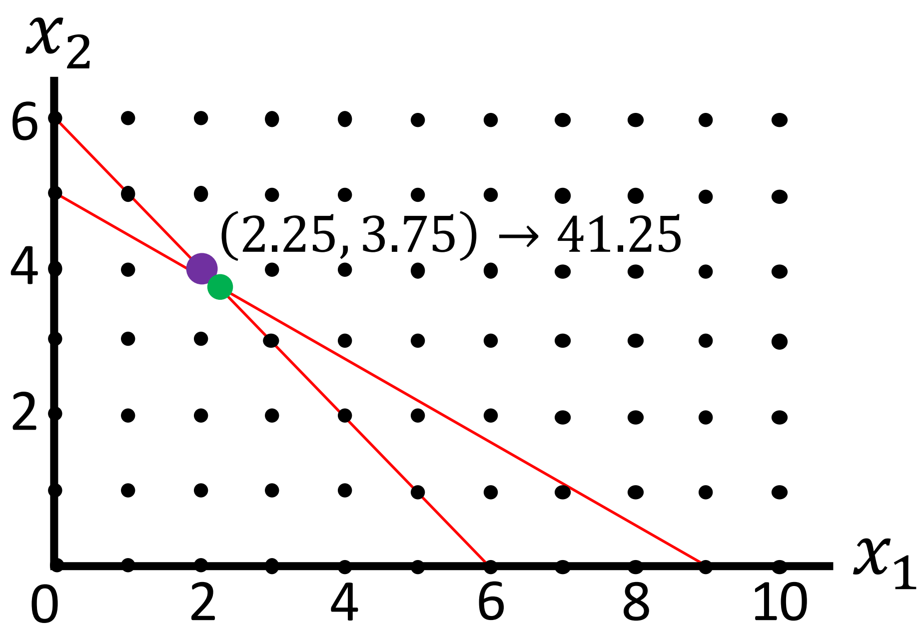
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution?**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

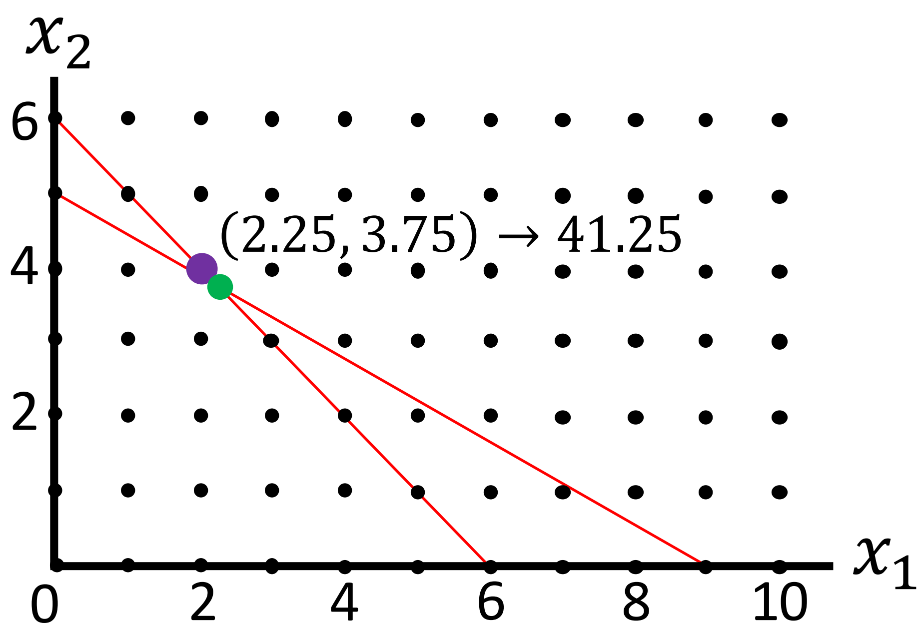
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution $\rightarrow$ optimal integer solution?

- **Closest integer solution? – Not feasible**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

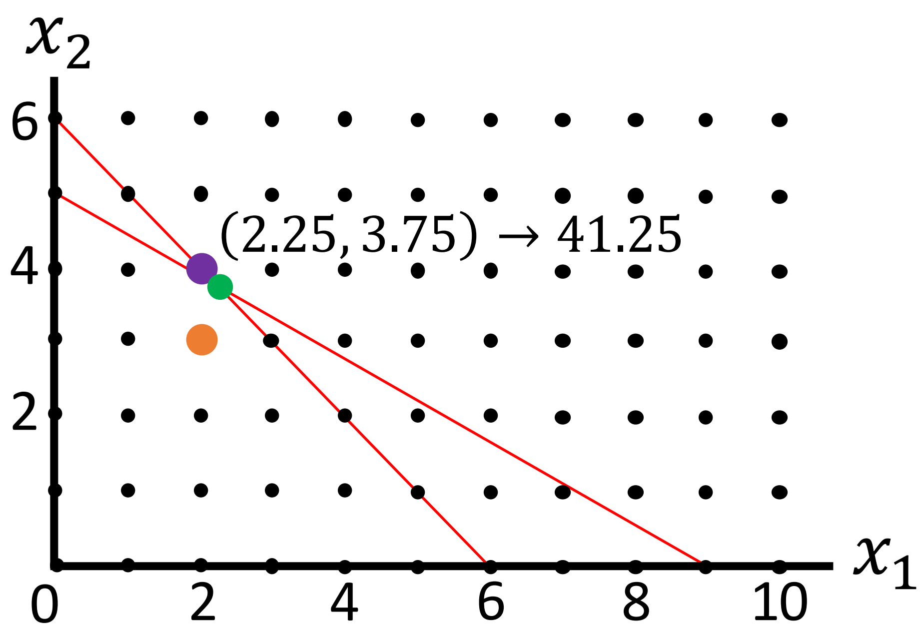
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution?**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

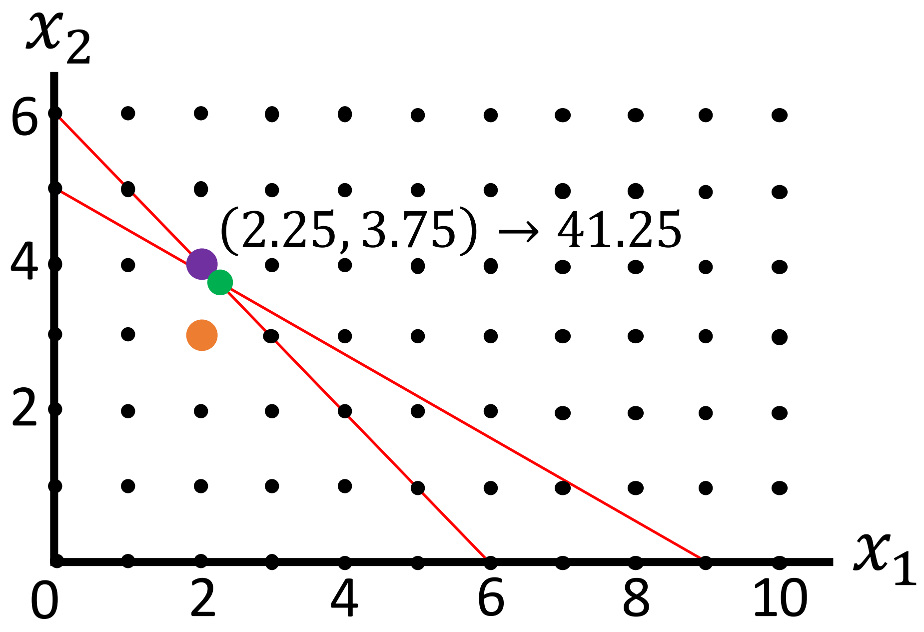
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution? – Obj = 34**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to:

$x_1 + x_2 \leq 6$

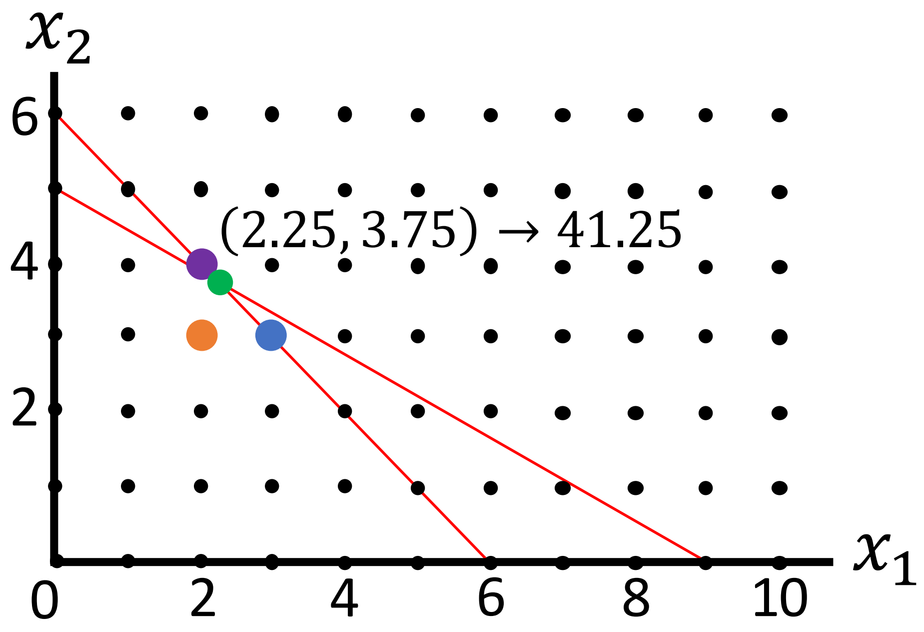
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution? – Obj = 34**
- **Closest feasible integer solution on feasible region boundary?**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

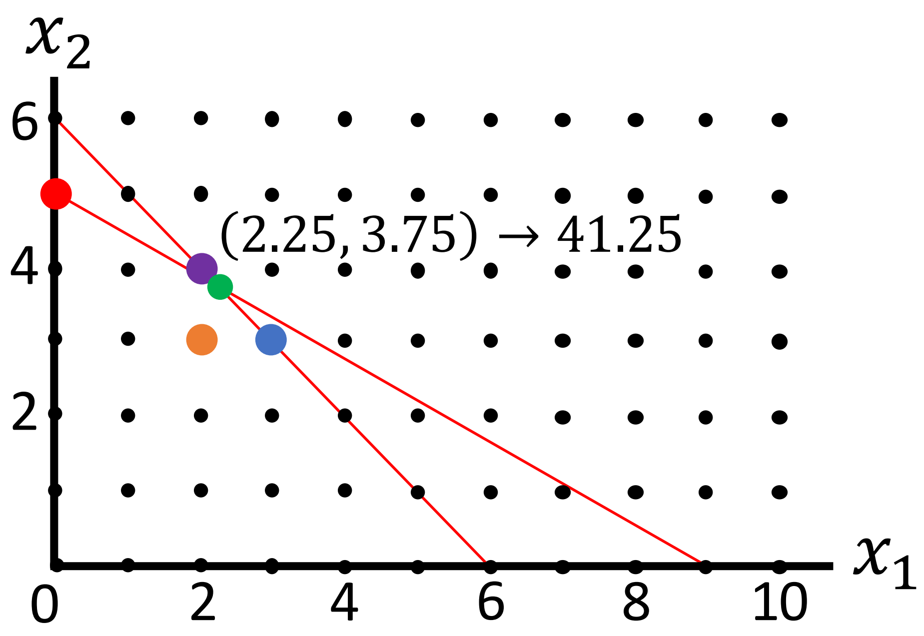
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution? – Obj = 34**
- **Closest feasible integer solution on feasible region boundary? – Obj = 39**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

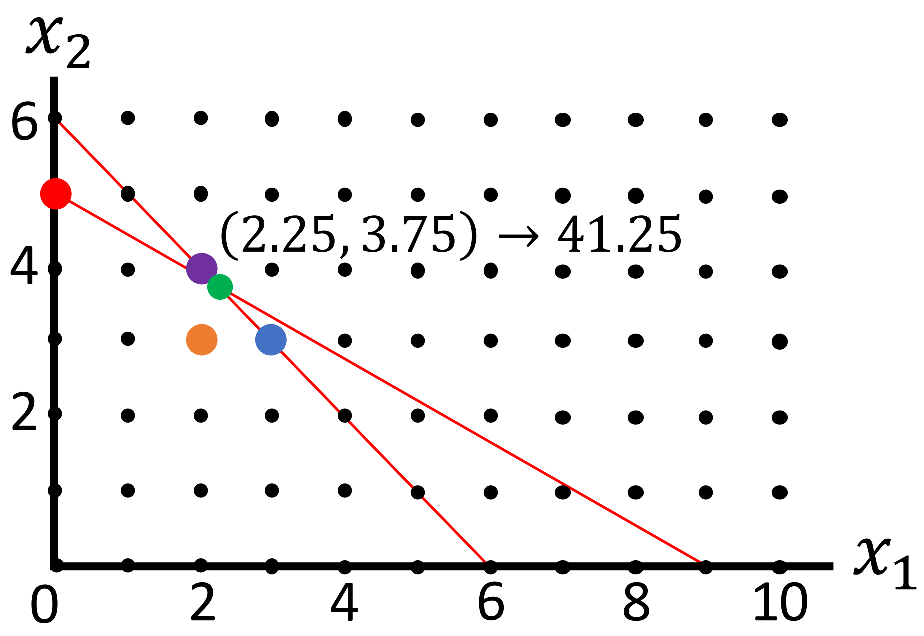
$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution? – Obj = 34**
- **Closest feasible integer solution on feasible region boundary? – Obj = 39**
- **Actual optimal – Obj = 40**

Why are ILPs hard to solve?



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

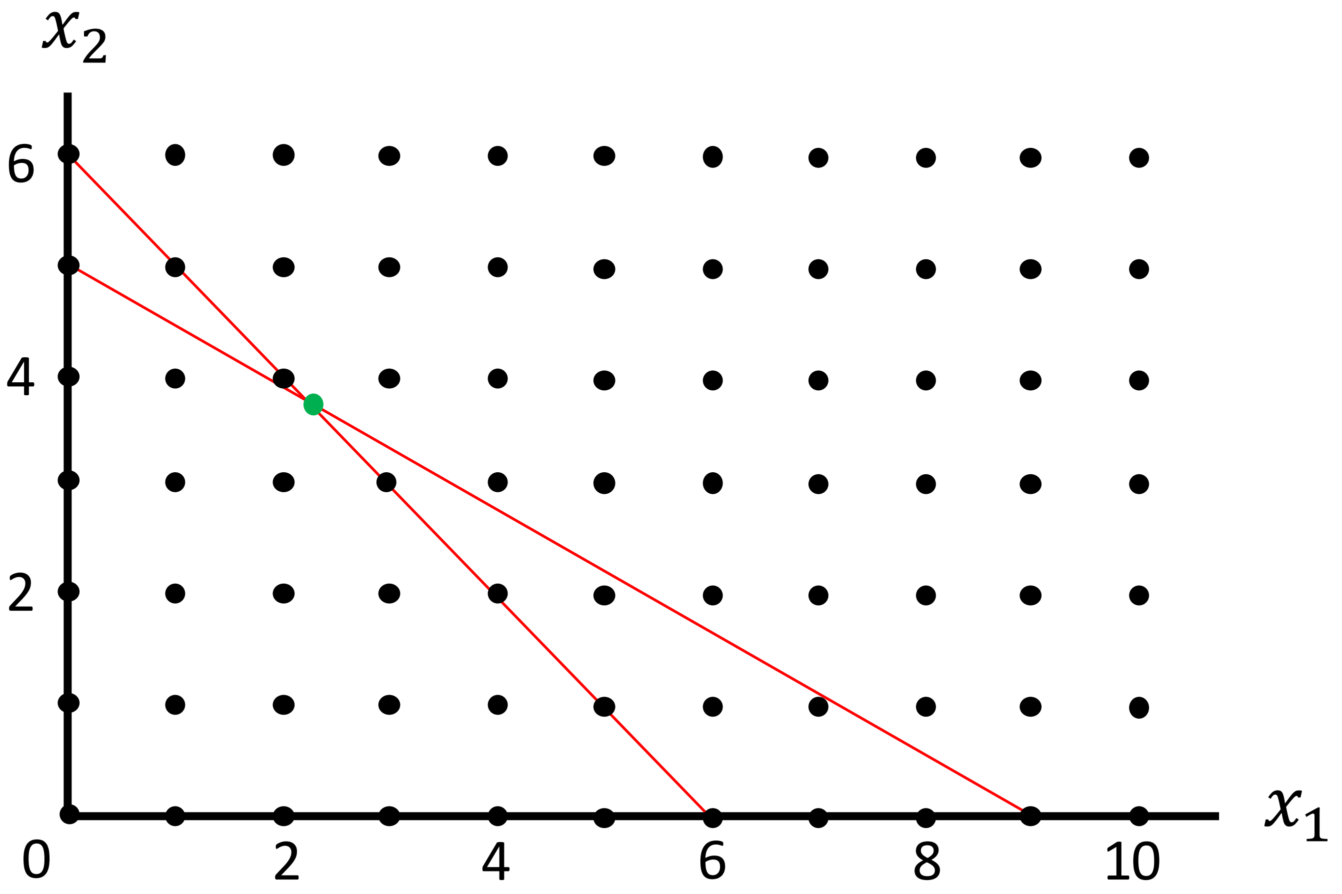
$x_1, x_2 \geq 0$

No guarantee that the optimal solution is on the feasible region boundary!

Optimal continuous solution → optimal integer solution?

- **Closest integer solution? – Not feasible**
- **Closest feasible integer solution? – Obj = 34**
- **Closest feasible integer solution on feasible region boundary? – Obj = 39**
- **Actual optimal – Obj = 40**

Why are ILPs hard to solve?



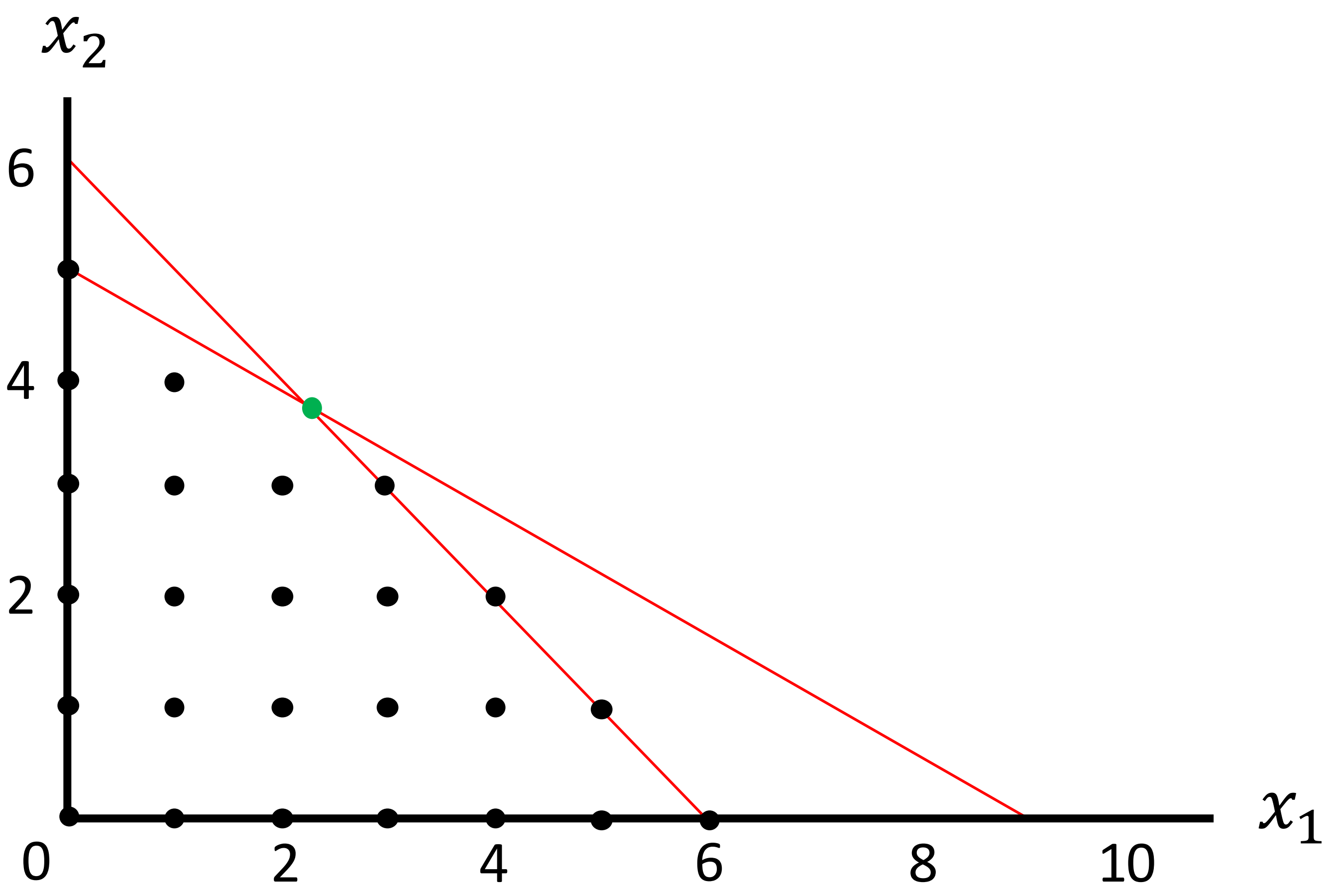
$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

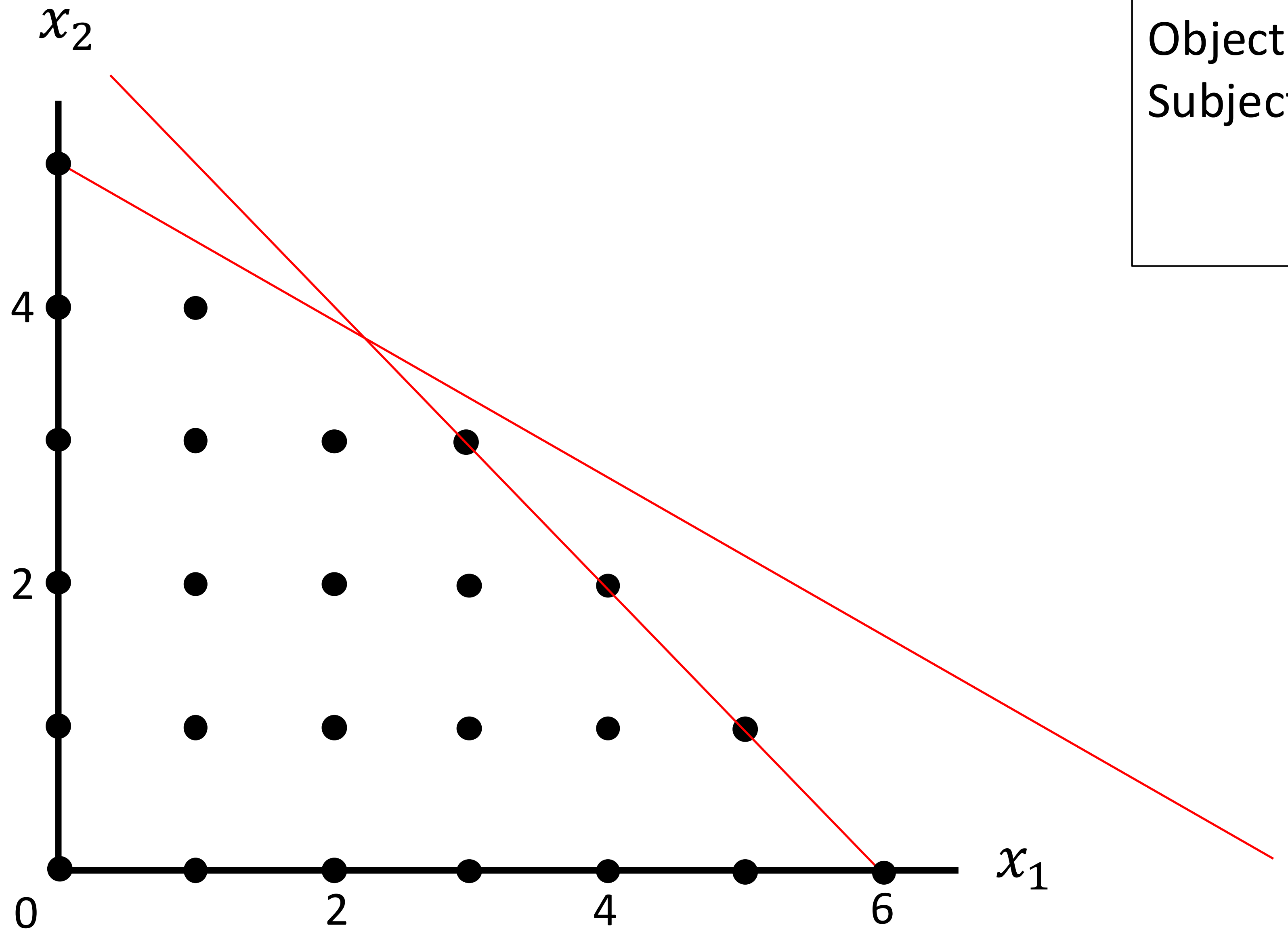
$x_1, x_2 \in \mathbb{N}$

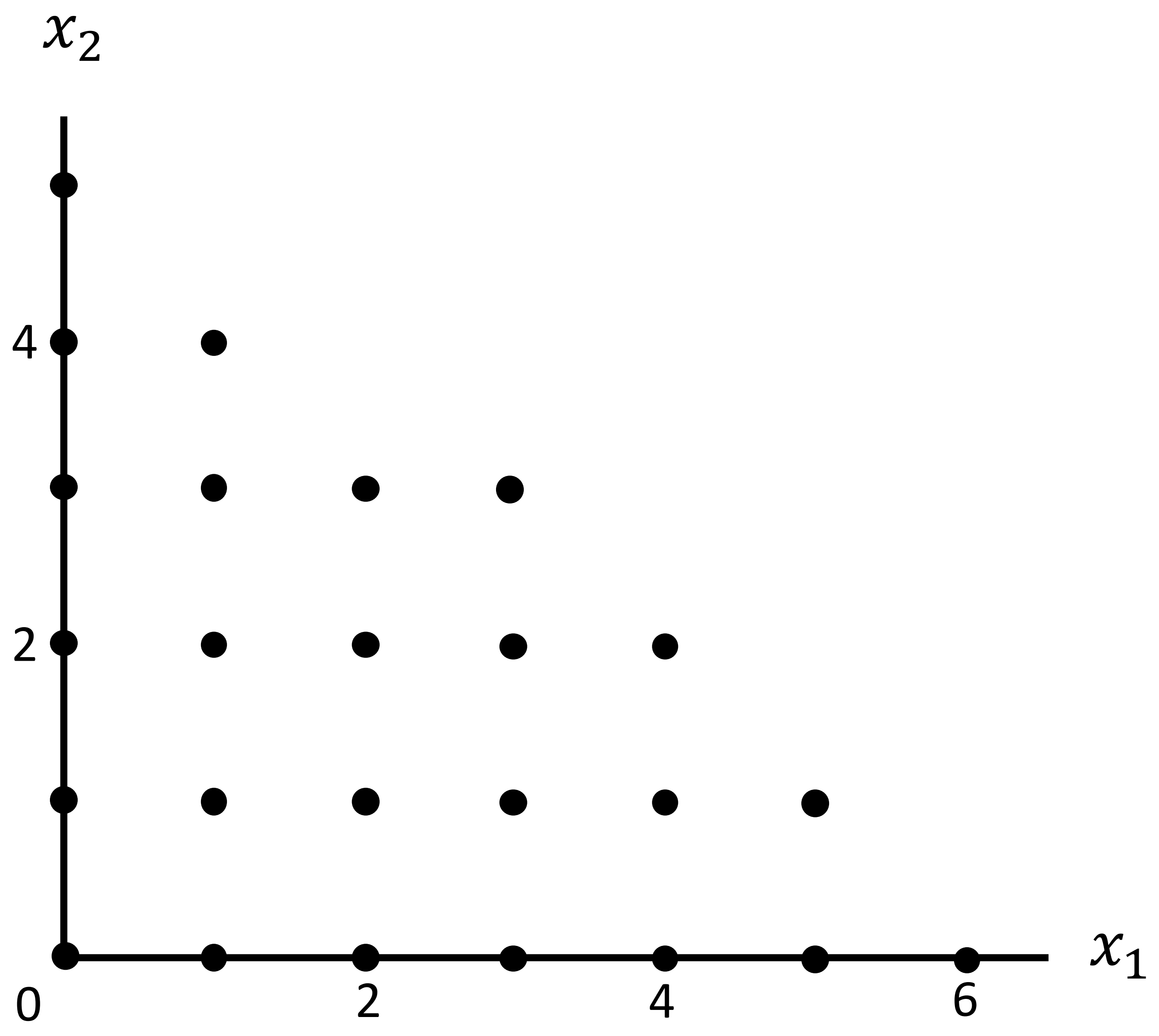
Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$





$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

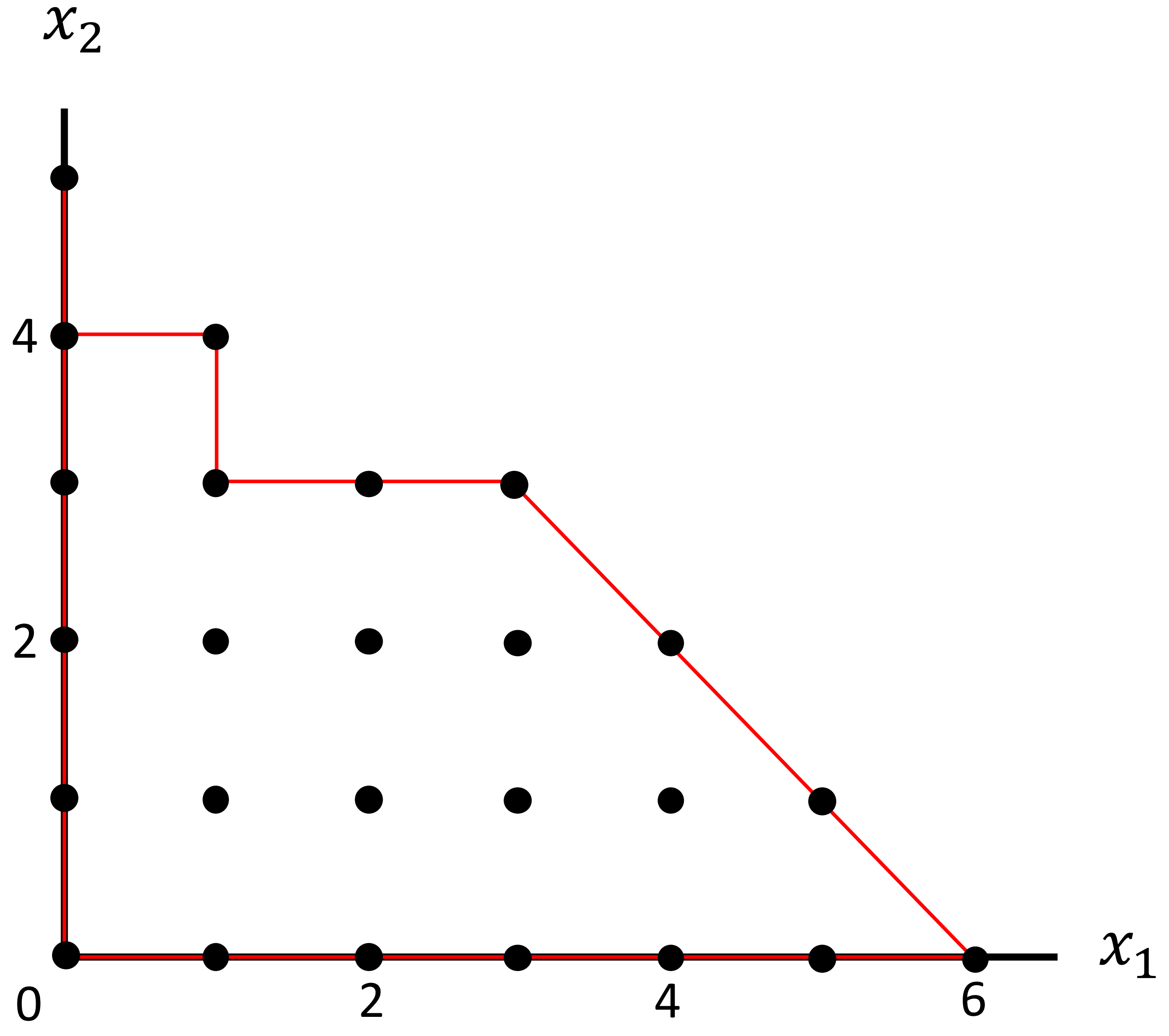
$5x_1 + 9x_2 \leq 45$

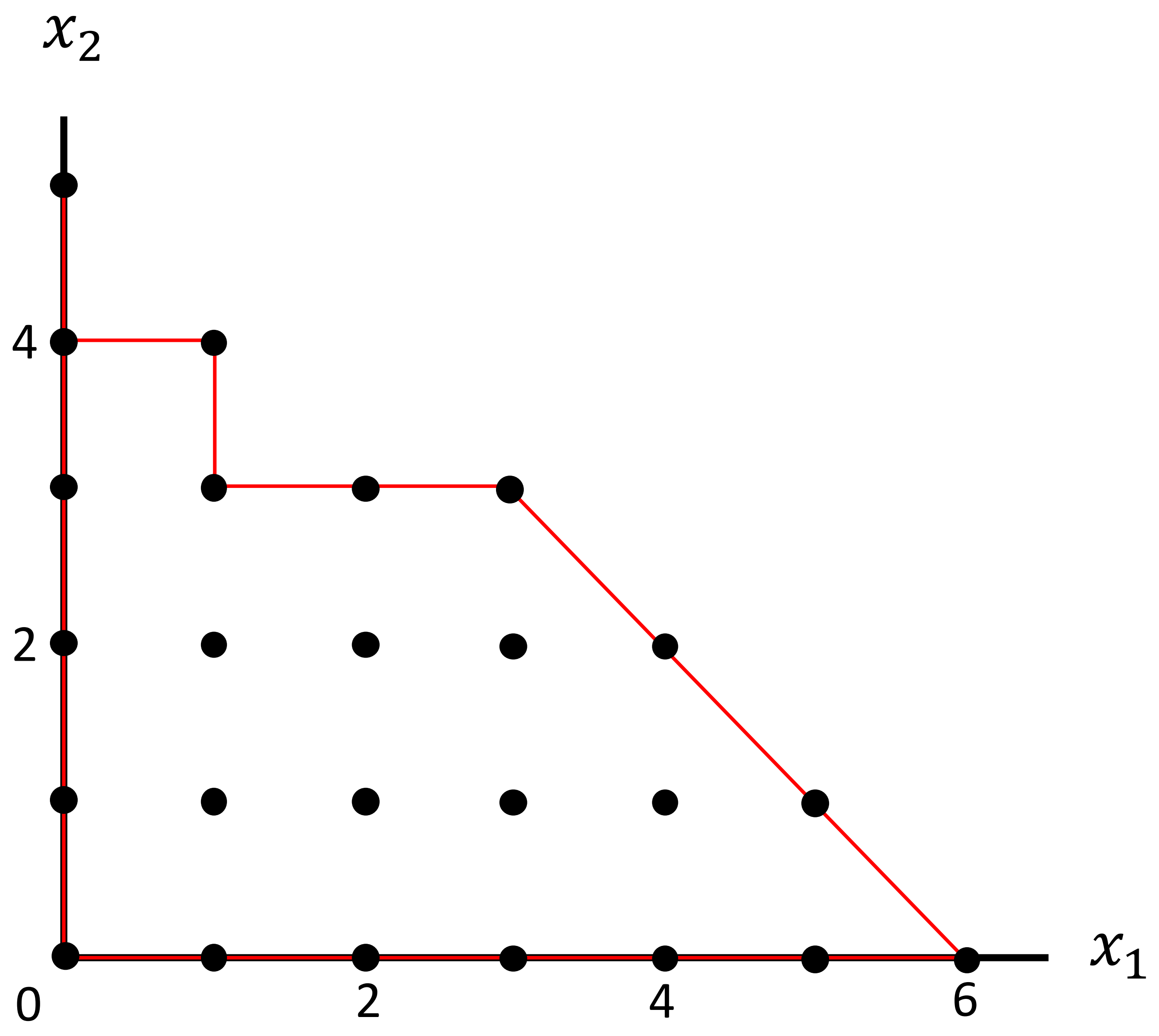
$x_1, x_2 \geq 0$

$x_1, x_2 \in \mathbb{N}$

Objective:	$\max 5x_1 + 8x_2$
Subject to:	$x_1 + x_2 \leq 6$
	$5x_1 + 9x_2 \leq 45$
	$x_1, x_2 \geq 0$

Integer feasible region:



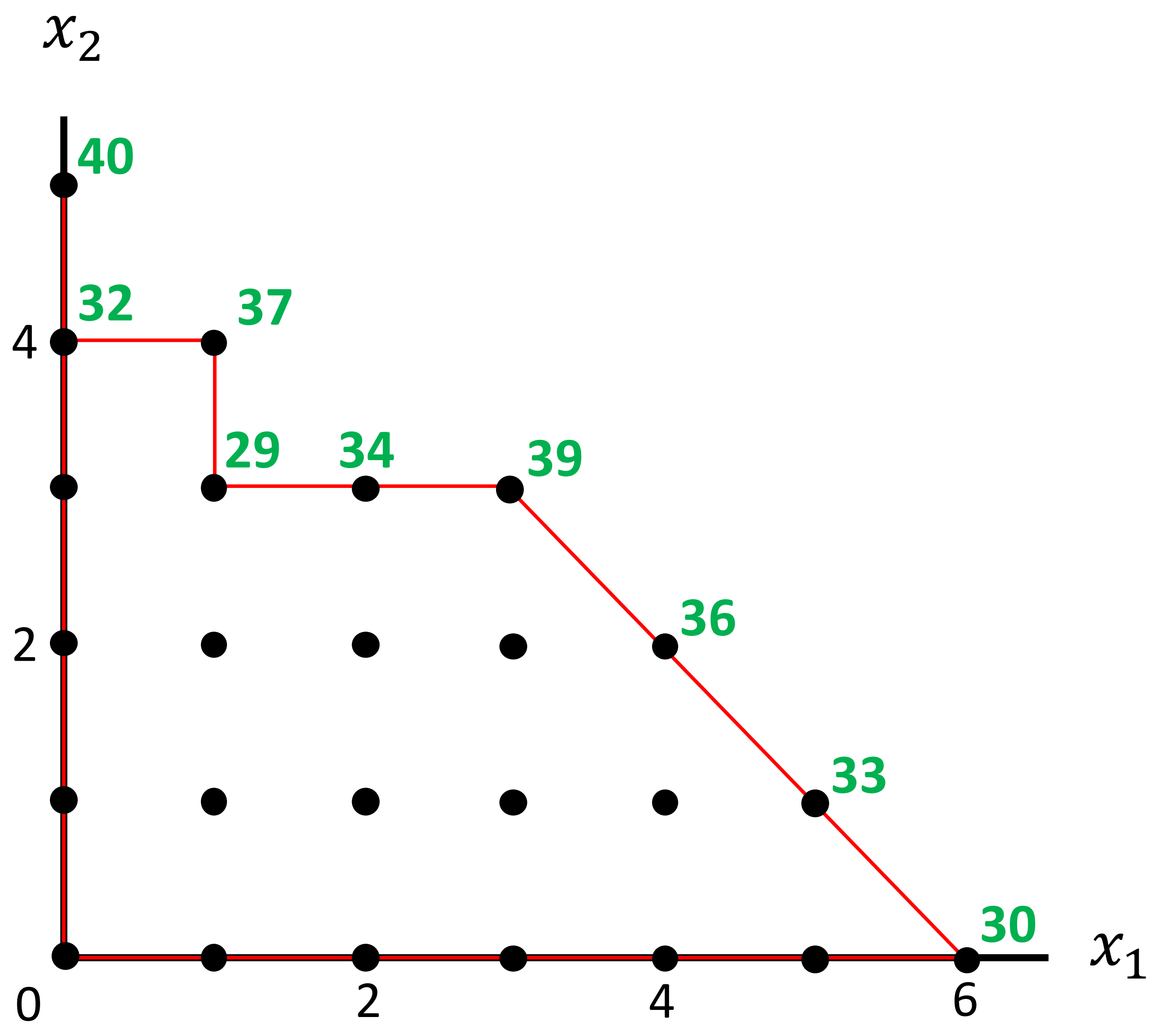


$x_1, x_2 \in \mathbb{N}$

Objective:	$\max 5x_1 + 8x_2$
Subject to:	$x_1 + x_2 \leq 6$
	$5x_1 + 9x_2 \leq 45$
	$x_1, x_2 \geq 0$

Integer feasible region:

- Not convex.



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

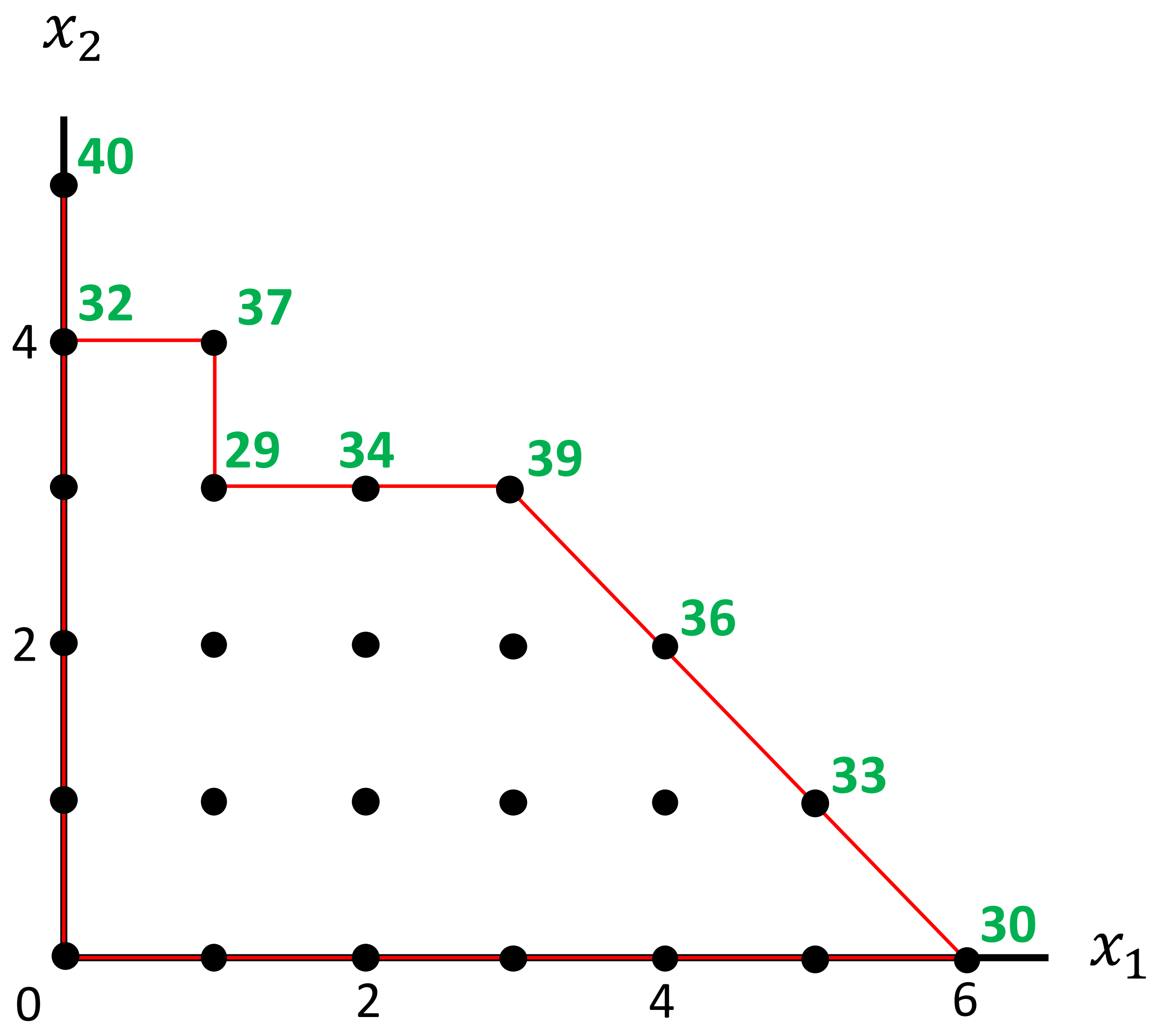
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Integer feasible region:

- Not convex.



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

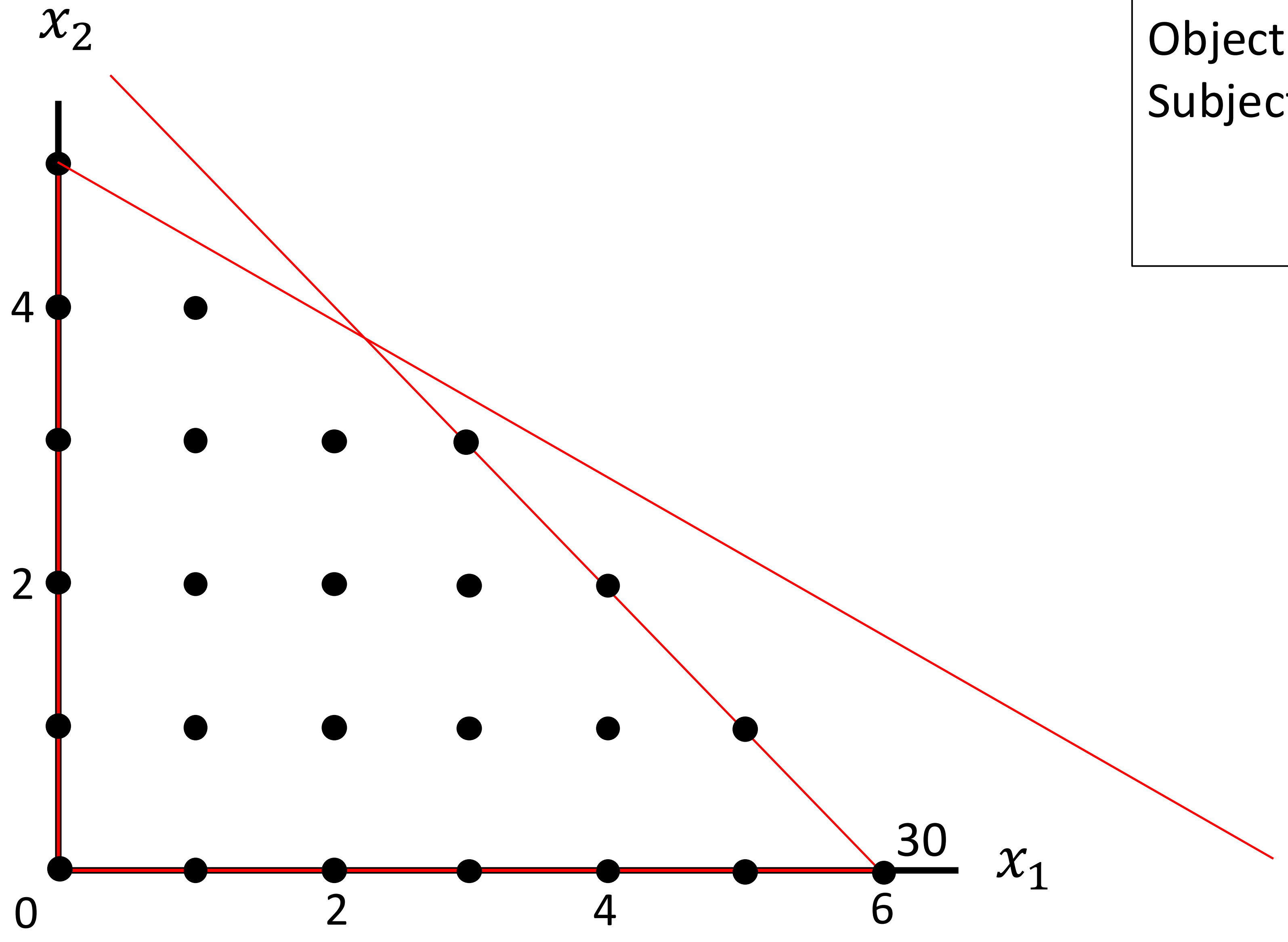
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Integer feasible region:

- Not convex.
- local optimum $\neq$ global optimum.



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to:
$x_1 + x_2 \leq 6$
$5x_1 + 9x_2 \leq 45$
$x_1, x_2 \geq 0$

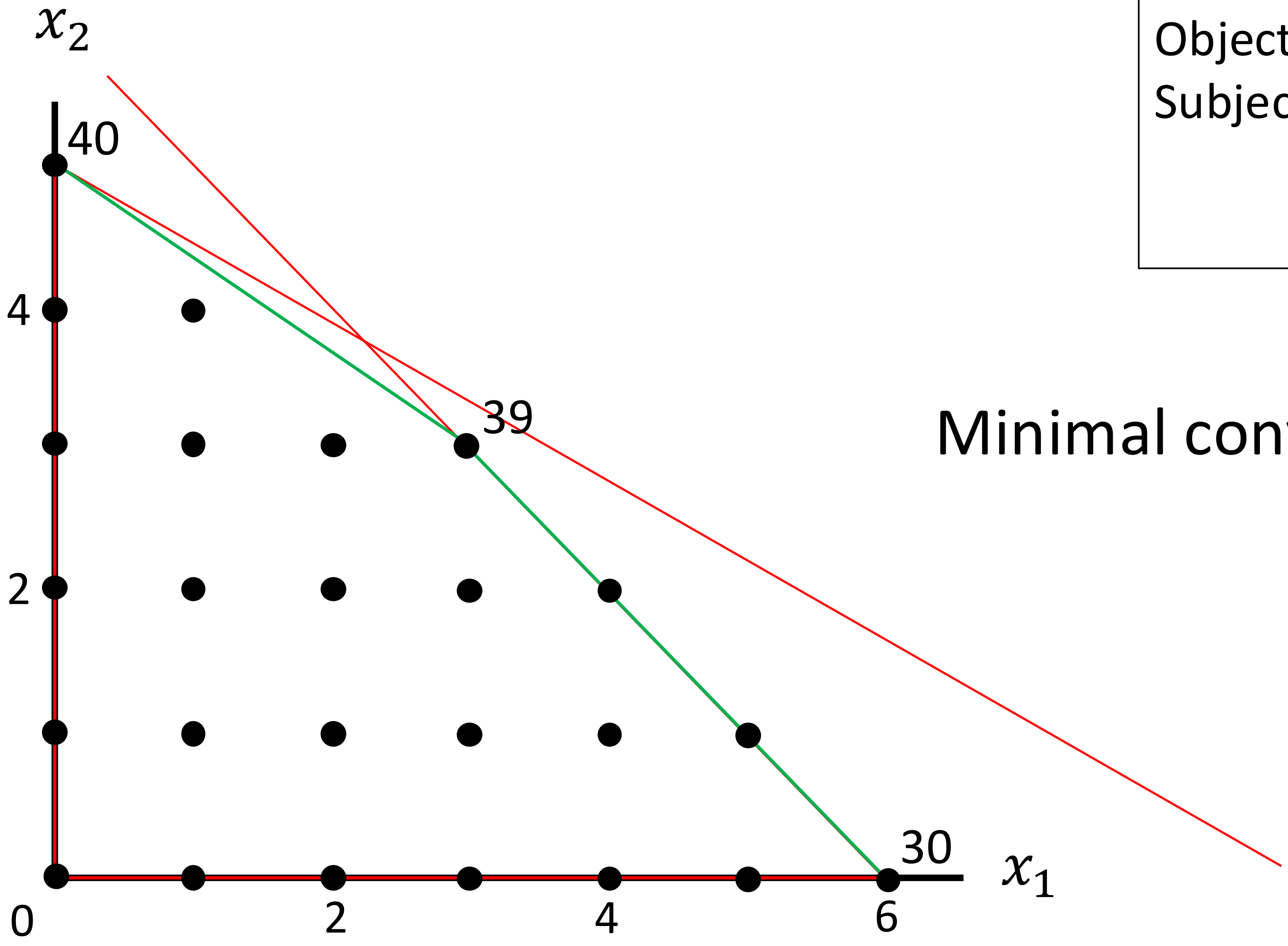
$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

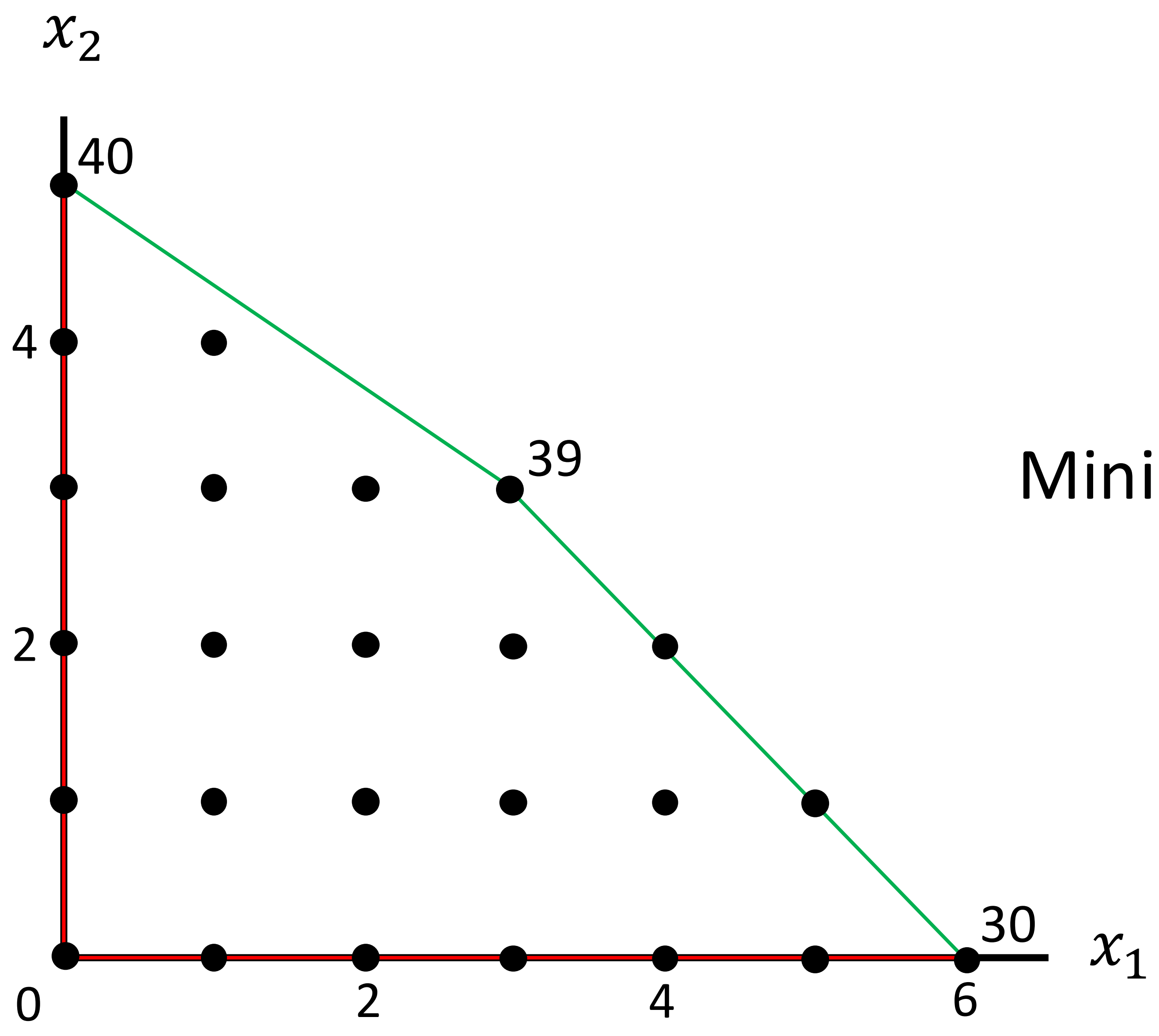
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$



Minimal convex hull (integer hull):



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

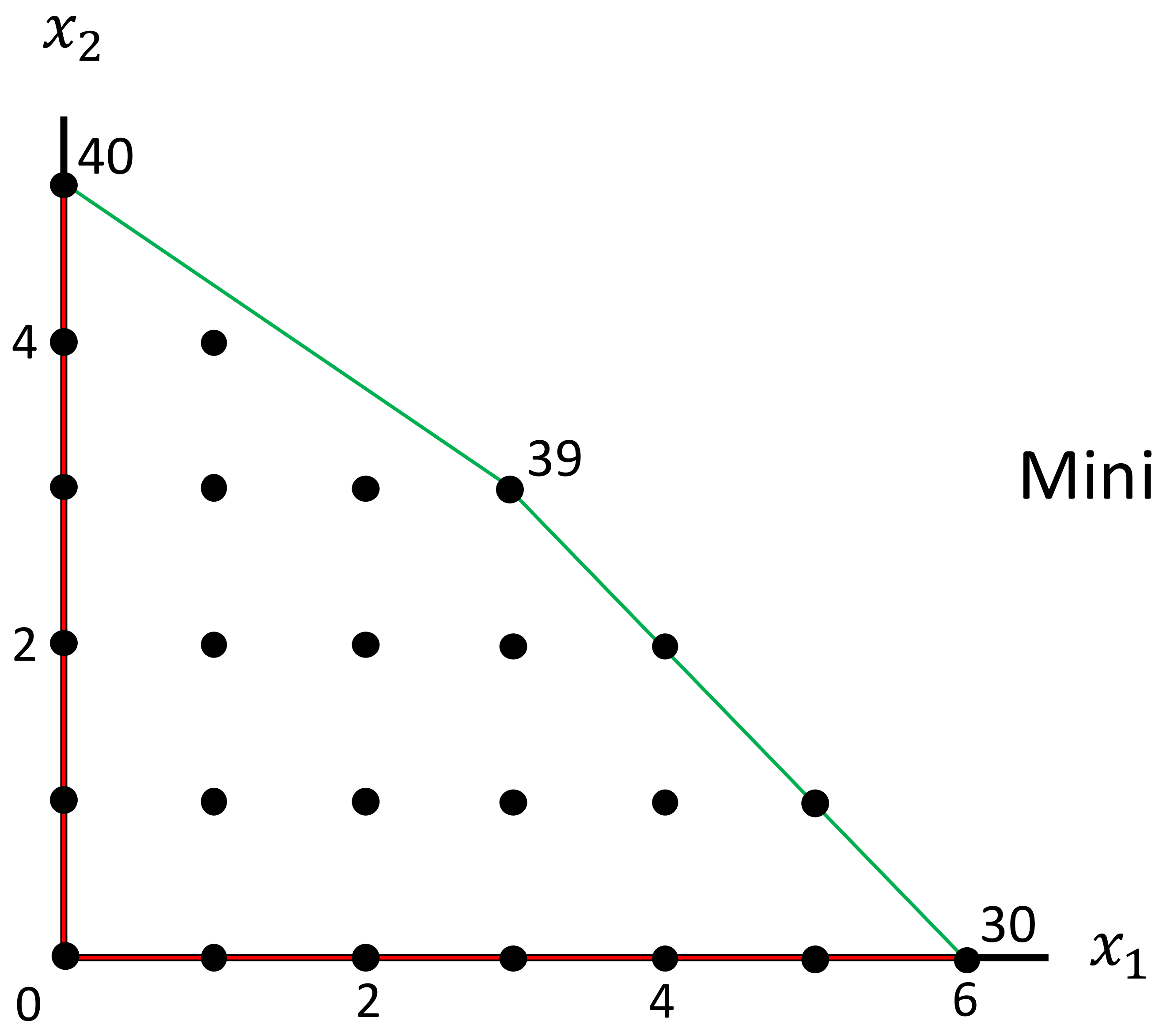
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Minimal convex hull (integer hull):

- Convex.



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

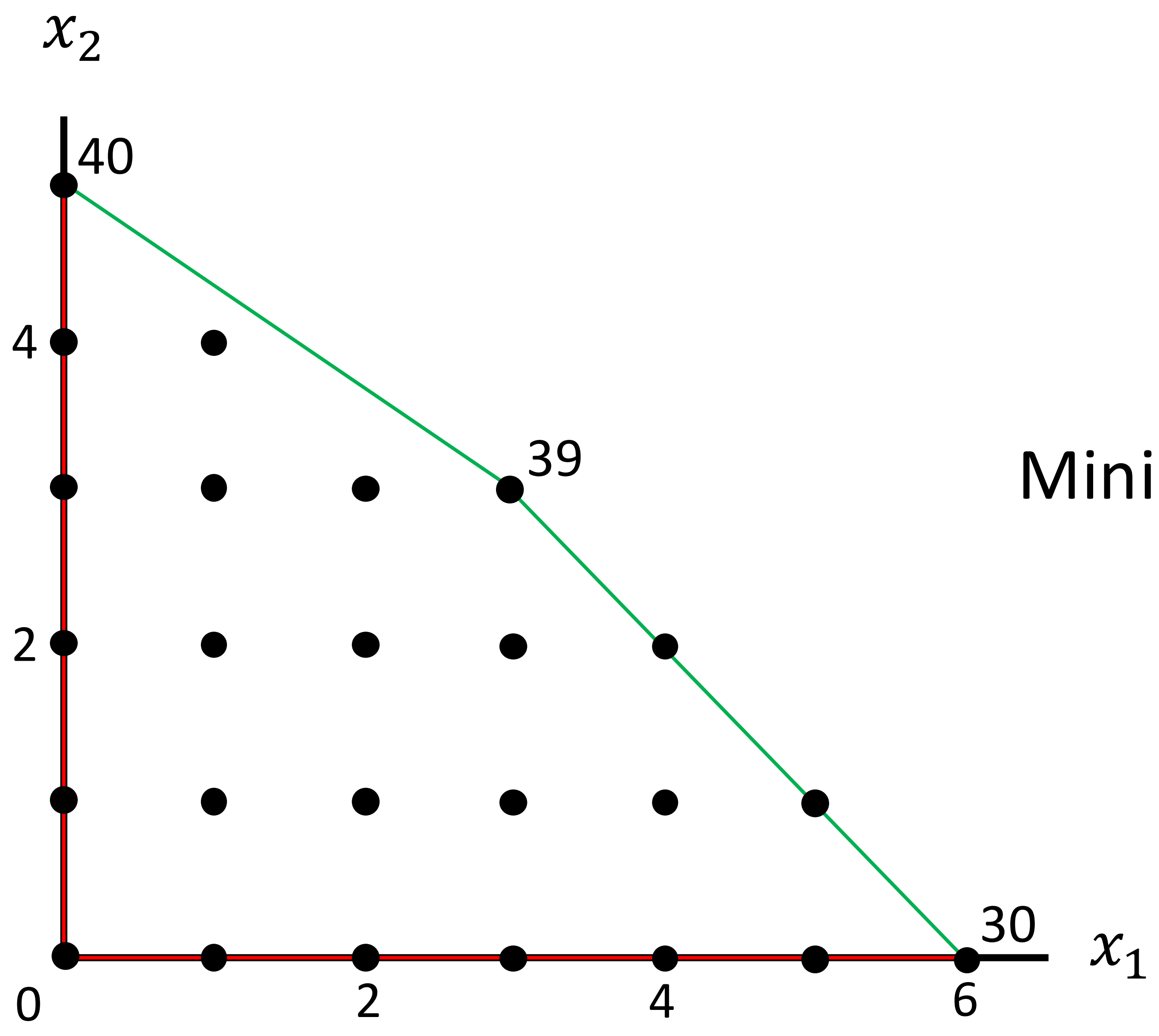
Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.



$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,

  $n = \#$ points

  $d = \#$ dimensions

If you had the integer hull, Simplex would easily find the optimum. Calculating the integer hull is usually harder than solving the ILP.

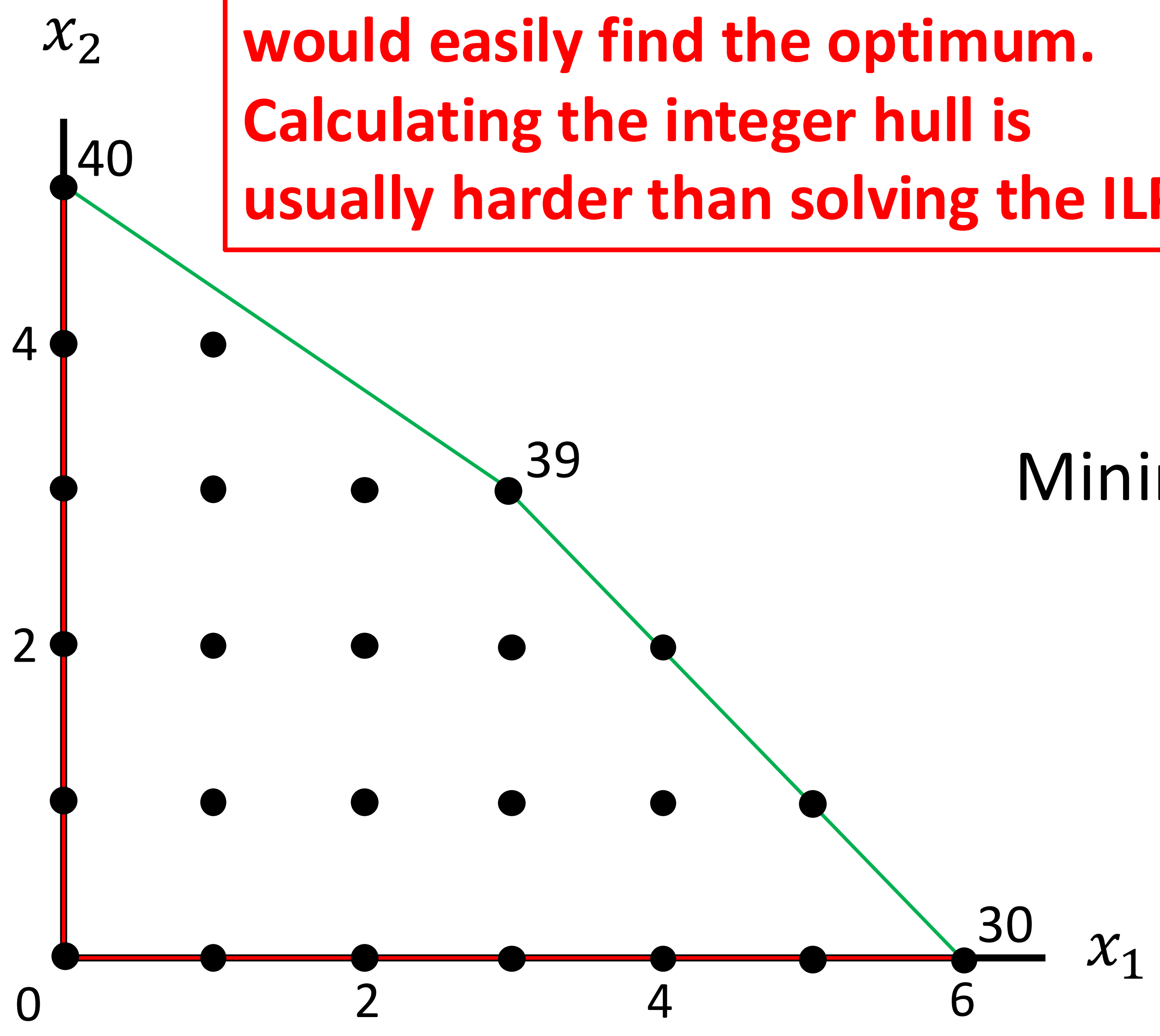
$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$



Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,

  $n = \#$ points

  $d = \#$ dimensions

Solving ILPs

How can we solve ILPs?

Solving ILPs

ILP

Maximization

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

Maximization

Solving ILPs

ILP → $\boldsymbol{x} \in \mathbb{Z} \to \boldsymbol{x} \in \mathbb{R}$ → **LP Relaxation**

1. Relax ILP

Maximization

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Solving ILPs

2. Solve relaxed LP.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

Maximization

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 7.4 with an integer solution (i.e., $x_i \in \mathbb{Z}, \forall i$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$x \in \mathbb{Z}$**, obj = 7.4**

Maximization

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 7.4 with an integer solution (i.e., $x_i \in \mathbb{Z}, \forall i$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{Z}$, **obj = 7.4**

Maximization

We've found the optimal!

Since the relaxed LP has **more options** to increase the objective value, $OPT_{ILP} \leq OPT_{LP}$ (for a maximization problem).

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

Maximization

UB = 100
LB = ?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1

subproblem 2

UB = 100
LB = ?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximization

4. Solve subproblem LPs.

subproblem 1

subproblem 2

UB = 100
LB = ?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = 100

LB = ?

subproblem 1

$x \in \mathbb{Z}$, **obj = 85**

subproblem 2

$x \in \mathbb{Z}$, **obj = 90**

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP — $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, obj = 100

subproblem 1
$\boldsymbol{x} \in \mathbb{Z}$, obj = 85

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, obj = 90

Optimal

UB = 90
LB = 90

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = 98
LB = 90

subproblem 1

$x \in \mathbb{R}$, **obj = 98**

subproblem 2

$x \in \mathbb{Z}$, **obj = 90**

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, obj = 100

UB = 98

LB = 90

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, obj = 98

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, obj = 90

So far, we have:

1. Feasible (integer) solution.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

ILP — $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

LP Relaxation: $\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1: $\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2: $\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

UB = 98
LB = 90

So far, we have:

1. Feasible (integer) solution.
2. Upper and lower bounds on optimal.

Solving ILPs

ILP

$x \in \mathbb{Z} \rightarrow x \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = 98
LB = 90

subproblem 1

$x \in \mathbb{R}$, obj = 98

subproblem 2

$x \in \mathbb{Z}$, obj = 90

So far, we have:

1. Feasible (integer) solution.
2. Upper and lower bounds on optimal.

Branch and Bound Plan: Use these to restrict the search space and identify optimality.

Solving ILPs

ILP

Maximization

UB = 98
LB = 90

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

subproblem 1.1

subproblem 1.2

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.